分析力学

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Chapter 1

Lagrange 力学

1.1 约束与广义坐标

1.1.1 约束的分类

在运动学中, 限制条件称为**约束 (constraint)**。

假如研究的物体对象由 n 个质点组成, 那么在三维空间中, 取第 i 个质点的位矢 $\mathbf{r}_i$, $i = 1, 2, \ldots, n$, 其正交分量为 x_i, y_i, z_i。

如果约束方程不显含时间, 即

$$f = f(\mathbf{r}_1, \ldots, \mathbf{r}_n), \tag{1.1.1}$$

则称其为**定常约束 (sceleronomous constraint)**或稳定约束; 如果约束方程显含时间, 即

$$f = f(t, \mathbf{r}_1, \ldots, \mathbf{r}_n), \tag{1.1.2}$$

则称其为**非定常约束 (rheonomous constraint)**或非稳定约束。

通俗的来讲, 定常约束不显含时间, 即 $\partial f/\partial t = 0$, 也就意味着约束方程不随时间变化, 是稳定的。而非定常约束则相反, 它是关于时间 t 的函数, 即 $\partial f/\partial t \neq 0$, 意味着它可能会变化, 是非稳定的。

接下来看看两个例子。如图 1.1 所示, 长为 l, 一端固定在 O 点的刚性杆, 因为它是刚性的, 不可缩短, 故其另一端点 (x, y, z) 有约束 $x^2 + y^2 + z^2 = l^2$。

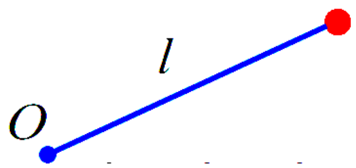

图 1.1: 一端固定在 O 点的刚性杆

再如图 1.2 所示, 长为 l, 一端固定在 O 点的不可伸长轻绳, 因其可以缩短, 故其另一端点 (x, y, z) 有约束 $x^2 + y^2 + z^2 \leq l^2$。

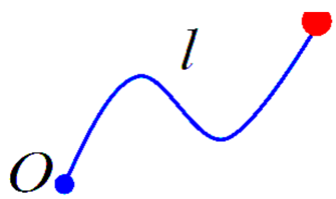

图 1.2: 一端固定在 O 点的轻绳

这两种情况就分别对应以下两种约束。

$$f(t, \mathbf{r}_1, \ldots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_n) = 0 \tag{1.1.3}$$

称为**双侧约束 (bilateral constraint)**, 等号限制住了系统, 是不可脱离的约束。而

$$f(t, \mathbf{r}_1, \ldots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_n) \geq 0 \tag{1.1.4}$$

称为**单侧约束 (unilateral constraint)**, 可以在某方面脱离约束。若单侧约束的形式为 $f \leq 0$, 则可以通过原约束两侧同乘以 -1 转化。

将

$$f(t, \mathbf{r}_1, \ldots, \mathbf{r}_n) = 0 \tag{1.1.5}$$

称为**几何约束 (geometric constraint)**, 将

$$f(t, \mathbf{r}_1, \ldots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_n) = 0 \tag{1.1.6}$$

称为**微分约束 (differential constraint)**或**运动约束 (kinematic constraint)**。

几何约束不显含速度, 而微分约束显含速度。将几何约束 $f(t, \mathbf{r}_i) = 0$ 两侧对时间 t 求导, 得

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \right) \\ &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} \right) = 0, \end{aligned} \tag{1.1.7}$$

经过求导后得到的式子显含速度, 于是几何约束就变为微分约束, 也就是说只要限制住了位置, 也就必然限制住了速度。于是, 微分约束就可以分成两类: **完整约束 (holonomic constraint)**和**非完整约束 (nonholonomic constraint)**。完整约束即可对时间积分的微分约束, 又称几何约束; 非完整约束即不可对时间积分的微分约束。

仅有完整约束的力学体系称为**完整系统 (holonomic system)**, 其余力学体系都称为**非完整系统 (nonholonomic system)**。

1.1.2 广义坐标

能够唯一确定系统位形的独立坐标, 称为**广义坐标 (generalized coordinate)**。广义坐标的量纲不一定是长度, 也可以是角度、质量、时间的量纲等。广义坐标的个数等于系统的自由度, 记为 s。对于 n 个质点组成的系统, 其自由度为 $3n$, 但是由于约束的存在, 系统的自由度会减少, 即 $s = 3n -$ 完整约束的个数 $< 3n$。

如图 1.3 所示, 摆长为 l 的单摆中, θ 就是广义坐标, 而也可以取 x 为广义坐标, 因为它们在这个体系中的值是唯一的。但是, (x, y) 和 y 都不是广义坐标, 因为存在单摆摆到同一高度 y 的情况。

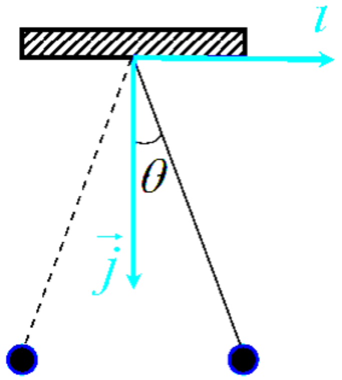

图 1.3: 单摆

Lagrange[1]首先用符号 q_α 表示第 α 个广义坐标, 其中 $\alpha = 1, 2, \ldots, s$。广义坐标对时间求导, 即得到 $\dot{q}_\alpha$, 称为**广义速度 (generalized velocity)**。

1.2 虚功原理

1.2.1 基本形式的虚功原理

我们知道, 对于 t 时刻的坐标矢量 $\mathbf{r}_i$, 其经过 t 时间, 有位移 $\mathrm{d}\mathbf{r}_i$, 即在 $t + \mathrm{d}t$ 时刻的坐标矢量为 $\mathbf{r}_i + \mathrm{d}\mathbf{r}_i$。若假定时间不变, 即 $\delta t = 0$, 则有约束允许的位移 $\delta\mathbf{r}_i$, 称**虚位移 (virtual displacement)**。对于定常约束 $(\partial f/\partial t = 0)$, 也就是说, 在时不变 (等时) 的情况下, 变分算符 δ 和微分算符 d 是等价的, 因此, 可以将虚位移 $\delta\mathbf{r}_i$ 看做是实位移 $\mathrm{d}\mathbf{r}_i$ 的特例。而对于非定常约束, 实位移和虚位移是不同的。

模仿 Newton 力学的功的定义, 有**虚功 (virtual work)**

$$\delta W = \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) \cdot \delta\mathbf{r}_i, \tag{1.2.1}$$

其中 $\mathbf{F}_i$ 是主动力, $\mathbf{R}_i$ 是约束力。满足

$$\sum_{i=1}^{n} \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0 \tag{1.2.2}$$

的约束称为**理想约束 (ideal constraint)**, 即约束力对虚位移所作的功为零。

进而, 满足理想约束下的虚功为

$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta\mathbf{r}_i. \tag{1.2.3}$$

令 $\delta W = 0$, 即得到**虚功原理 (principle of virtual work)**

$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0. \tag{1.2.4}$$

[1]拉格朗日, Joseph-Louis Lagrange, 原名 Giuseppe Luigi Lagrangia, 1736.01.25—1813.04.10, 意大利数学家、物理学家和天文学家, 后归化为法国人。

命题 1.1
系统在约束允许的平衡位形下维持平衡的充要条件是

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0. \tag{1.2.5}$$

1.2.2 广义坐标下的虚功原理

从基本形式的虚功原理出发, 可以推导出广义坐标下的虚功原理。知道 $\mathbf{r}_i = \mathbf{r}_i(t, q)$, 求微分得

$$\mathrm{d}\mathbf{r}_i = \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \mathrm{d}q_\alpha + \frac{\partial \mathbf{r}_i}{\partial t} \mathrm{d}t. \tag{1.2.6}$$

将微分改为变分, 时刻牢记 $\delta t = 0$, 即得到虚位移

$$\delta \mathbf{r}_i = \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha + \cancel{\frac{\partial \mathbf{r}_i}{\partial t} \delta t} = \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha. \tag{1.2.7}$$

代入式 (1.2.4) 中, 得广义坐标下的虚功

$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha = \sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right) \delta q_\alpha. \tag{1.2.8}$$

再令 $\delta q_\alpha = 0$, 即得到广义坐标下的虚功原理

$$\boxed{\sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = 0}. \tag{1.2.9}$$

若广义坐标有位移的量纲, 那么上式就有力的量纲, 将其称为**广义力 (generalized force)**, 记为 Q_α, 即

$$Q_\alpha = \sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}. \tag{1.2.10}$$

注意, 如果广义坐标有角度的量纲, 那么广义力就有力矩的量纲; 广义坐标有体积的量纲, 那么广义力就有压强的量纲。如此看来, 广义力并不是真正的力, 而是一种推广的概念。

1.2.3 保守系统下的虚功原理

记势能为 V, 对于保守系统, 有

$$\mathbf{F}_i = -\nabla_i V = -\left(\frac{\partial V}{\partial x_i} \mathbf{i} + \frac{\partial V}{\partial y_i} \mathbf{j} + \frac{\partial V}{\partial z_i} \mathbf{k} \right), \tag{1.2.11}$$

将式 (1.2.11) 和 $\mathbf{r}_i = x_i \mathbf{i} + y_i \mathbf{j} + z_i \mathbf{k}$ 代入式 (1.2.10) 中, 得

$$Q_\alpha = \sum_{i=1}^{n} - \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_\alpha} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_\alpha} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_\alpha} \right)$$

$$= -\frac{\partial V}{\partial q_\alpha} = 0, \tag{1.2.12}$$

即得到保守系统下的平衡条件

$$\boxed{\frac{\partial V}{\partial q_\alpha} = 0}. \tag{1.2.13}$$

1.3 Euler-Lagrange 方程

1.3.1 d'Alembert 原理

在 Newton 力学中, 对于保守系统, 其运动方程为

$$\mathbf{F}_i + \mathbf{R}_i = m_i \ddot{\mathbf{r}}_i, \quad i = 1, 2, \ldots, n \tag{1.3.1}$$

其中 $\mathbf{F}_i$ 称为主动力, $\mathbf{R}_i$ 称为约束力 (受外界约束的被动力)。不妨将 $m_i \ddot{\mathbf{r}}_i$ 移到等号左侧, 即将其看做一项力, 得

$$\mathbf{F}_i + \mathbf{R}_i - m_i \ddot{\mathbf{r}}_i = 0, \tag{1.3.2}$$

其中 $-m_i \ddot{\mathbf{r}}_i$ 称为 d'Alembert 惯性力, 或反向有效力。这就是 **d'Alembert 原理 (d'Alembert's principle)**, 或称 **Lagrange-d'Alembert 原理 (Lagrange-d'Alembert principle)**。这时候, 动力学问题就成为了主动力、约束力和反向有效力共同作用下的平衡问题。

模仿式 (1.2.8) 的推导过程, 忽略约束力, 得

$$\begin{aligned} 0 &= \sum_{i=1}^{n} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha \\ &= \sum_{\alpha=1}^{s} \left[\sum_{i=1}^{n} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right] \delta q_\alpha \\ &= \sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} - \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right) \delta q_\alpha \\ &= \sum_{\alpha=1}^{s} \left(Q_\alpha - \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right) \delta q_\alpha. \end{aligned} \tag{1.3.3}$$

就得到一个新的动力学方程

$$Q_\alpha - \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = 0, \quad \alpha = 1, 2, \ldots, s. \tag{1.3.4}$$

在下一小节会接着研究这个方程。

1.3.2 第一类 Lagrange 方程

先证明两个公式。

性质 1.1

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha}. \tag{1.3.5}$$

证明. $\mathbf{r}_i$ 对时间求导, 有

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial t} + \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha, \tag{1.3.6}$$

可以看出 $\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i\left(t, q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s\right)$, 在后文把 $q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s$ 分别记为 $q, \dot{q}$。

将 $\partial \mathbf{r}_i/\partial q_\alpha$ 对时间求导, 得

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right) + \sum_{\beta=1}^{s} \frac{\partial}{\partial q_\beta}\left(\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right)\dot{q}_\beta \\
&= \frac{\partial}{\partial q_\alpha}\left(\frac{\partial \mathbf{r}_i}{\partial t}\right) + \frac{\partial}{\partial q_\alpha}\sum_{\beta=1}^{s}\left(\frac{\partial \mathbf{r}_i}{\partial q_\beta}\right)\dot{q}_\beta \\
&= \frac{\partial}{\partial q_\alpha}\underbrace{\left(\frac{\partial \mathbf{r}_i}{\partial t} + \sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\beta\right)}_{\dot{\mathbf{r}}_i} \\
&= \frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha}.
\end{aligned} \tag{1.3.7}$$

□

性质 1.2

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_\alpha} = \frac{\partial \mathbf{r}_i}{\partial q_\alpha}. \tag{1.3.8}$$

证明.

$$\begin{aligned}
\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_\alpha} &= \frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{\partial \mathbf{r}_i}{\partial t} + \sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\beta\right) \\
&= \cancel{\frac{\partial}{\partial \dot{q}_\alpha}\frac{\partial \mathbf{r}_i}{\partial t}} + \frac{\partial}{\partial \dot{q}_\alpha}\left(\sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\beta\right) \\
&= \sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\beta}\frac{\partial \dot{q}_\beta}{\partial \dot{q}_\alpha} = \sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\beta}\delta_{\beta\alpha} \\
&= \frac{\partial \mathbf{r}_i}{\partial q_\alpha}.
\end{aligned} \tag{1.3.9}$$

□

因为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right) = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} + \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right), \tag{1.3.10}$$

所以,式 (1.3.4) 可以写成

$$Q_\alpha - \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right) + \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right) = 0, \tag{1.3.11}$$

再代入式 (1.3.5) 和 (1.3.8), 得

$$Q_\alpha - \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_\alpha}\right) + \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha} = 0. \tag{1.3.12}$$

还知道动能

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i, \tag{1.3.13}$$

则可进一步得到

$$\boxed{Q_\alpha - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) + \frac{\partial T}{\partial q_\alpha} = 0}, \tag{1.3.14}$$

这就是基本形式的 Lagrange 方程, 叫做**第一类 Lagrange 方程 (Lagrange's equation of the first kind)**。其中 $\partial T/\partial \dot{q}_\alpha$ 叫做**广义动量 (generalized momentum)**。

1.3.3 Euler-Lagrange 方程

若主动力是保守力, 即 $Q_\alpha = -\partial V/\partial q_\alpha$, 则根据式 (1.3.14), 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} + \frac{\partial V}{\partial q_\alpha} = 0 \tag{1.3.15}$$

$$\Longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial (T-V)}{\partial q_\alpha} = 0. \tag{1.3.16}$$

又因为势能 $V = V(t, q)$, 不显含广义速度, 所以 $\partial V/\partial \dot{q}_\alpha = 0$, 于是

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial (T-V)}{\partial \dot{q}_\alpha}\right] - \frac{\partial (T-V)}{\partial q_\alpha} = 0. \tag{1.3.17}$$

记 $L = T - V$, 则有

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0}, \tag{1.3.18}$$

其中 L 叫做 **Lagrangian**。这就是**第二类 Lagrange 方程 (Lagrange's equation of the second kind)**, 也叫 **Euler-Lagrange 方程 (Euler-Lagrange equation)**, 现在得到了保守系的 Lagrange 方程。

可以与 Newton 力学进行对比,式 (1.3.18) 的物理意义一目了然。

1.3.4 可略坐标与运动常数

若 Lagrangian L 不显含某一广义坐标 q_α, 即 $\partial L/\partial q_\alpha = 0$, 则称 q_α 为**可略坐标 (ignorable coordinate)**[2]。由式 (1.3.18), 可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0, \tag{1.3.19}$$

对时间积分后, 得守恒的广义动量

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \text{常数}, \tag{1.3.20}$$

这称为**运动常数 (constant of motion)**。只要是系统的运动的守恒量, 都可以称为运动常数。

1.3.5 广义能量

由式 (1.3.14), 可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}, \tag{1.3.21}$$

对每一项乘上广义速度 $\dot{q}_\alpha$ 并求和, 得

$$\sum_{\alpha=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)\right]\dot{q}_\alpha - \sum_{\alpha=1}^{s}\frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha = -\sum_{\alpha=1}^{s}\frac{\partial V}{\partial q_\alpha}\dot{q}_\alpha, \tag{1.3.22}$$

又因为

$$\sum_{\alpha=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right)\right] = \sum_{\alpha=1}^{s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha + \sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot{q}_\alpha}\ddot{q}_\alpha, \tag{1.3.23}$$

代入式 (1.3.22) 中, 有

$$\sum_{\alpha=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right)\right] - \sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot{q}_\alpha}\ddot{q}_\alpha - \sum_{\alpha=1}^{s}\frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha = -\sum_{\alpha=1}^{s}\frac{\partial V}{\partial q_\alpha}\dot{q}_\alpha. \tag{1.3.24}$$

现在来看动能 $T = T(t, q, \dot{q})$ 对时间的导数

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial t} + \sum_{\alpha=1}^{s}\frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot{q}_\alpha}\ddot{q}_\alpha, \tag{1.3.25}$$

和势能 $V = V(t, q)$ 对时间的导数

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \sum_{\alpha=1}^{s}\frac{\partial V}{\partial q_\alpha}\dot{q}_\alpha, \tag{1.3.26}$$

[2] 有些地方称为可遗坐标或循环坐标, 我认为是不恰当的。

联立式 (1.3.24)—(1.3.26), 得

$$\sum_{\alpha=1}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right)-\frac{\mathrm{d}T}{\mathrm{d}t}+\frac{\partial T}{\partial t}=-\frac{\mathrm{d}V}{\mathrm{d}t}+\frac{\partial V}{\partial t}, \tag{1.3.27}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha-T+V\right)=\frac{\partial (V-T)}{\partial t}=-\frac{\partial L}{\partial t}, \tag{1.3.28}$$

又 V 不显含 $\dot{q}_\alpha$, 广义动量 $p_\alpha=\frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial T}{\partial \dot{q}_\alpha}$, 所以

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L\right)=-\frac{\partial L}{\partial t}. \tag{1.3.29}$$

定义广义能量

$$\boxed{H=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L}, \tag{1.3.30}$$

则

$$\frac{\mathrm{d}H}{\mathrm{d}t}=-\frac{\partial L}{\partial t}. \tag{1.3.31}$$

若 L 不显含时间 t, 即 $\frac{\partial L}{\partial t}=0$, 则

$$\frac{\mathrm{d}H}{\mathrm{d}t}=0, \tag{1.3.32}$$

此时 H 为常数。

如果 $\mathbf{r}_i=\mathbf{r}_i(t,q)$ 显含时间, 有动能

$$\begin{aligned}
T&=\frac{1}{2}\sum_{i=1}^{n}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i=\frac{1}{2}\sum_{i=1}^{n}m_i\left(\frac{\partial \mathbf{r}_i}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha\right)\cdot\left(\frac{\partial \mathbf{r}_i}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha\right)\\
&=\frac{1}{2}\sum_{i=1}^{n}m_i\left(\frac{\partial \mathbf{r}_i}{\partial t}\cdot\frac{\partial \mathbf{r}_i}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha\cdot\frac{\partial \mathbf{r}_i}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial \mathbf{r}_i}{\partial t}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha+\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha\cdot\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\beta\right)\\
&=\frac{1}{2}\sum_{i=1}^{n}m_i\left(\frac{\partial \mathbf{r}_i}{\partial t}\cdot\frac{\partial \mathbf{r}_i}{\partial t}+2\sum_{\alpha=1}^{s}\frac{\partial \mathbf{r}_i}{\partial t}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha+\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta\right)\\
&=\frac{1}{2}\sum_{i=1}^{n}m_i\frac{\partial \mathbf{r}_i}{\partial t}\cdot\frac{\partial \mathbf{r}_i}{\partial t}+m_i\sum_{\alpha=1}^{s}\frac{\partial \mathbf{r}_i}{\partial t}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\dot{q}_\alpha+\frac{1}{2}m_i\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta.
\end{aligned} \tag{1.3.33}$$

上式分别为广义动能的零次、一次和二次项, 记为 T_0,T_1,T_2, 则有 $T=T_0+T_1+T_2$。T_0 和 T_1 是由非定常约束带来的。

引入一个定理。

引理 1.1 Euler 齐次函数定理

设 $f(x_1,\dots,x_n)$ 是 n 个变量的 m 次齐次函数, 即

$$f(\lambda x_1,\dots,\lambda x_n)=\lambda^m f(x_1,\dots,x_n)\,, \tag{1.3.34}$$

则有

$$\sum_{i=1}^{n} x_i\frac{\partial f}{\partial x_i}=mf. \tag{1.3.35}$$

这就是 **Euler 齐次函数定理 (Euler's homogeneous function theorem)**。

根据 Euler 齐次函数定理, 有

$$\sum_{\alpha=1}^{s}\frac{\partial T_n}{\partial \dot q_\alpha}\dot q_\alpha=nT_n, \tag{1.3.36}$$

由此可得

$$\sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot q_\alpha}\dot q_\alpha=\sum_{\alpha=1}^{s}\frac{\partial(T_0+T_1+T_2)}{\partial \dot q_\alpha}\dot q_\alpha=0T_0+1T_1+2T_2=T_1+2T_2. \tag{1.3.37}$$

所以

$$\begin{aligned}H=\sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot q_\alpha}\dot q_\alpha-T+V&=(T_1+2T_2)-(T_0+T_1+T_2)+V\\&=T_2-T_0+V.\end{aligned} \tag{1.3.38}$$

如果 $\mathbf{r}_i=\mathbf{r}_i(q)$ 不显含时间, 那么式 (1.3.33) 只剩下二次项, 变为

$$T=\frac{1}{2}\sum_{i=1}^{n}m_i\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot q_\alpha\dot q_\beta, \tag{1.3.39}$$

由 Euler 齐次函数定理, 可得

$$\sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot q_\alpha}\dot q_\alpha=2T, \tag{1.3.40}$$

所以

$$H=2T-T+V=T+V. \tag{1.3.41}$$

当坐标不显含时间时, 广义能量 H 就是系统的机械能。

1.3.6 Euler-Lagrange 方程的应用

可以运用 Euler-Lagrange 方程求球坐标系下的加速度分量。在球坐标系下, 有 $\mathrm{d}r=\mathrm{d}r\mathbf{e}_r+r\,\mathrm{d}\theta\mathbf{e}_\theta+r\sin\theta\,\mathrm{d}\phi\mathbf{e}_\phi$, 则 $\mathbf{v}=\mathrm{d}r/\mathrm{d}t=\dot r\mathbf{e}_r+r\dot\theta\mathbf{e}_\theta+r\sin\theta\dot\phi\mathbf{e}_\phi$。取三个广义坐标 $q_r=r, q_\theta=$

$\theta, q_\phi = \phi$, 由式 (1.3.14), 可得

$$\begin{aligned}\delta W &= \mathbf{F}\cdot\delta\mathbf{r}\\ &= \left(F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi\right)\cdot\left(\delta r\mathbf{e}_r + r\delta\theta\mathbf{e}_\theta + r\sin\theta\delta\phi\mathbf{e}_\phi\right)\\ &= F_r\delta r + F_\theta r\delta\theta + F_\phi r\sin\theta\delta\phi\\ &= Q_r\delta q_r + Q_\theta\delta q_\theta + Q_\phi\delta q_\phi.\end{aligned} \tag{1.3.42}$$

即得广义力、力与加速度的关系

$$\begin{cases} Q_r = F_r = ma_r,\\ Q_\theta = F_\theta r = ma_\theta r,\\ Q_\phi = F_\phi r\sin\theta = ma_\phi r\sin\theta.\end{cases} \tag{1.3.43}$$

又因为动能

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right), \tag{1.3.44}$$

分别代入式 (1.3.14) 中, 可求得各广义力

$$\begin{aligned}Q_r &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{r}}\right) - \frac{\partial T}{\partial r}\\ &= m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}) - \frac{m}{2}\frac{\partial}{\partial r}\left(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)\\ &= m\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2\right),\end{aligned} \tag{1.3.45}$$

$$\begin{aligned}Q_\theta &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta}\\ &= m\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) - \frac{m}{2}\frac{\partial}{\partial\theta}\left(r^2\sin^2\theta\dot{\phi}^2\right)\\ &= mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2\right),\end{aligned} \tag{1.3.46}$$

$$\begin{aligned}Q_\phi &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi}\\ &= m\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\sin^2\theta\dot{\phi}\right) - 0\\ &= mr\sin\theta\left(2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}\right).\end{aligned} \tag{1.3.47}$$

得到各加速度分量

$$\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2,\\ a_\theta = \ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2,\\ a_\phi = 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}.\end{cases} \tag{1.3.48}$$

1.4 对称性

当 Lagrangian 具有某些特殊形式时, 会导致广义动量、能量的守恒, 这其实反映了系统的**对称性 (symmetry)**, 即在某种**变换 (transformation)**下, 系统的性质保持**不变性 (invariance)**。这种变换称为**对称变换 (symmetry transformation)**。

1.4.1 Noether 定理

设 Lagrangian $L(t,q,\dot{q})$ 描述一封闭系统, 设无穷小变换

$$t \to \bar{t} = t + \sum_{i=1}^{r} \epsilon_i \xi_i\left(\mathbf{t}, q(t)\right), \tag{1.4.1}$$

$$q_\alpha \to \bar{q}_\alpha = q_\alpha + \sum_{i=1}^{r} \epsilon_i \eta_{\alpha i}\left(\mathbf{t}, q(t)\right), \tag{1.4.2}$$

其中 ϵ_i 是无穷小参数, $\xi_i, \eta_{\alpha i}$ 是无穷小函数。当 $\epsilon_i = 0$ 时, $\bar{t} = t, \bar{q}_\alpha = q_\alpha$, 即无穷小变换退化为恒等变换。设变换后的 Lagrangian 为 $\overline{L}\left(\bar{t},\bar{q},\dot{\bar{q}}\right)$, 则有不变的作用量

$$S = \int_{t_1}^{t_2} L(t,q,\dot{q})\,\mathrm{d}t = \int_{\bar{t}_1}^{\bar{t}_2} \overline{L}\left(\bar{t},\bar{q},\dot{\bar{q}}\right)\mathrm{d}\bar{t} = \overline{S}, \tag{1.4.3}$$

又

$$\int_{\bar{t}_1}^{\bar{t}_2} \overline{L}\left(\bar{t},\bar{q},\dot{\bar{q}}\right)\mathrm{d}\bar{t} = \int_{t_1}^{t_2} \overline{L}\left(\bar{t},\bar{q},\dot{\bar{q}}\right)\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\,\mathrm{d}t, \tag{1.4.4}$$

所以

$$\overline{L}\left(\bar{t},\bar{q},\dot{\bar{q}}\right)\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = L(t,q,\dot{q}). \tag{1.4.5}$$

如果

$$\overline{L}(t,q,\dot{q}) = L(t,q,\dot{q}), \tag{1.4.6}$$

则称 L 在变换下具有**不变性 (invariance)**。如果

$$\overline{L}(t,q,\dot{q}) = L(t,q,\dot{q}) + \frac{\mathrm{d}\Lambda(t,q)}{\mathrm{d}t}, \tag{1.4.7}$$

则称 L 在变换下具有**协变性 (covariance)**。具有这两种性质的变换称为**对称变换 (symmetry transformation)**。

将式 (1.4.7) 代入式 (1.4.5), 得

$$\left[L\left(\bar{t},\bar{q},\dot{\bar{q}}\right) + \frac{\mathrm{d}\Lambda\left(\bar{t},\bar{q}\right)}{\mathrm{d}\bar{t}}\right]\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = L(t,q,\dot{q}). \tag{1.4.8}$$

保留一阶小量, 由式 (1.4.1) 可得

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = 1 + \sum_{i=1}^{r} \epsilon_i \frac{\mathrm{d}\xi_i}{\mathrm{d}t}, \tag{1.4.9}$$

$$\begin{aligned}\dot{\bar{q}}_\alpha &= \frac{\mathrm{d}\bar{q}_\alpha}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}\bar{q}_\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\bar{t}} \\ &= \left[\dot{q}_\alpha + \sum_{i=1}^{r}\epsilon_i\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t}\right]\left(1+\sum_{i=1}^{r}\epsilon_i\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)^{-1} \\ &\approx \left[\dot{q}_\alpha + \sum_{i=1}^{r}\epsilon_i\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t}\right]\left(1-\sum_{i=1}^{r}\epsilon_i\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right) \\ &\approx \dot{q}_\alpha + \sum_{i=1}^{r}\epsilon_i\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t} - \dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right).\end{aligned} \tag{1.4.10}$$

有

$$\begin{aligned}L\left(\bar{t},\bar{q},\dot{\bar{q}}\right) &= L(t,q,\dot{q}) + \frac{\partial L}{\partial t}\left(\bar{t}-t\right) + \sum_{\alpha=1}^{s}\left[\frac{\partial L}{\partial q_\alpha}\left(\bar{q}_\alpha - q_\alpha\right) + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\dot{\bar{q}}_\alpha - \dot{q}_\alpha\right)\right] \\ &= L(t,q,\dot{q}) + \frac{\partial L}{\partial t}\sum_{i=1}^{r}\epsilon_i\xi_i + \sum_{\alpha=1}^{s}\sum_{i=1}^{r}\epsilon_i\left[\frac{\partial L}{\partial q_\alpha}\eta_{\alpha i} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t} - \dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)\right],\end{aligned} \tag{1.4.11}$$

$$\begin{aligned}\frac{\mathrm{d}\Lambda\left(\bar{t},\bar{q}\right)}{\mathrm{d}\bar{t}} &= \frac{\mathrm{d}\Lambda\left(\bar{t},\bar{q}\right)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\bar{t}} \\ &\approx \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \Lambda}{\partial t}\left(\bar{t}-t\right) + \sum_{\alpha=1}^{s}\frac{\partial \Lambda}{\partial q_\alpha}\left(\bar{q}_\alpha - q_\alpha\right)\right]\left(1-\sum_{i=1}^{r}\epsilon_i\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right) \\ &\approx \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{r}\epsilon_i\left[\frac{\partial \Lambda}{\partial t}\xi_i + \sum_{\alpha=1}^{s}\frac{\partial \Lambda}{\partial q_\alpha}\eta_{\alpha i}\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{r}\epsilon_i\Pi_i(t,q),\end{aligned} \tag{1.4.12}$$

其中 $\Pi_i(t,q) = \dfrac{\partial \Lambda}{\partial t}\xi_i + \displaystyle\sum_{\alpha=1}^{s}\frac{\partial \Lambda}{\partial q_\alpha}\eta_{\alpha i}$。把上两式代入式 (1.4.8) 中, 得

$$\begin{aligned}L(t,q,\dot{q}) + \frac{\partial L}{\partial t}\sum_{i=1}^{r}\epsilon_i\xi_i + \sum_{\alpha=1}^{s}\sum_{i=1}^{r}\epsilon_i\left[\frac{\partial L}{\partial q_\alpha}\eta_{\alpha i} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t} - \dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)\right] \\ + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{r}\epsilon_i\Pi_i(t,q) = \left(1-\sum_{i=1}^{r}\epsilon_i\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)L(t,q,\dot{q}),\end{aligned} \tag{1.4.13}$$

整理得

$$\frac{\partial L}{\partial t}\xi_i + \sum_{\alpha=1}^{s}\left[\frac{\partial L}{\partial q_\alpha}\eta_{\alpha i} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t} - \dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)\right] + L\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = -\frac{\mathrm{d}\Pi_i}{\mathrm{d}t}, \tag{1.4.14}$$

对于给定的 L 和变换, 计算上式等号左边的表达式, 若可以表示为某一函数对时间的全微分, 则

变换是对称变换, 同时能够得到 $\Pi_i(t,q)$ 的表达式。又

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t,q,\dot{q})=\frac{\partial L}{\partial t}+\sum_{\alpha=1}^{s}\left(\frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha\right), \tag{1.4.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)=\sum_{\alpha=1}^{s}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha}\right)(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)+\frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t}-\ddot{q}_\alpha\xi_i-\dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)\right], \tag{1.4.16}$$

代入式 (1.4.14) 中, 消去 $\partial L/\partial t$, 得

$$\frac{\mathrm{d}L}{\mathrm{d}t}\xi_i-\sum_{\alpha=1}^{s}\left(\frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha\xi_i+\frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha\xi_i\right)+\sum_{\alpha=1}^{s}\left[\frac{\partial L}{\partial q_\alpha}\eta_{\alpha i}+\frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t}-\dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right)\right]+L\frac{\mathrm{d}\xi_i}{\mathrm{d}t}+\frac{\mathrm{d}\Pi_i}{\mathrm{d}t}=0 \tag{1.4.17}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}(L\xi_i+\Pi_i)+\sum_{\alpha=1}^{s}\left[\frac{\partial L}{\partial q_\alpha}(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)+\frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\eta_{\alpha i}}{\mathrm{d}t}-\dot{q}_\alpha\frac{\mathrm{d}\xi_i}{\mathrm{d}t}-\ddot{q}_\alpha\xi_i\right)\right]=0 \tag{1.4.18}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)+L\xi_i+\Pi_i\right]+\sum_{\alpha=1}^{s}\left[\frac{\partial L}{\partial q_\alpha}(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha}\right)(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)\right]=0 \tag{1.4.19}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)+L\xi_i+\Pi_i\right]+\sum_{\alpha=1}^{s}\left(\frac{\partial L}{\partial q_\alpha}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha}\right)(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)=0. \tag{1.4.20}$$

当系统运动满足 Euler-Lagrange 方程时, 由上式知, 存在运动常数

$$I_i(t,q)=\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)+L\xi_i+\Pi_i, \tag{1.4.21}$$

又

$$\frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial}{\partial \dot{q}_\alpha}\left(\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-H\right)=p_\alpha, \tag{1.4.22}$$

所以

$$\begin{aligned}I_i(t,q)&=\sum_{\alpha=1}^{s}p_\alpha(\eta_{\alpha i}-\dot{q}_\alpha\xi_i)+\left(\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-H\right)\xi_i+\Pi_i\\&=\sum_{\alpha=1}^{s}\eta_{\alpha i}p_\alpha-H\xi_i+\Pi_i,\end{aligned} \tag{1.4.23}$$

这就是 **Noether 定理 (Noether's theorem)**[3]。

考虑变换形式不变, 则有 $\Lambda=0$。如果仅作时间平移变换 $\eta_{\alpha i}=0$, 则有 $I_i=-H\xi_i$。如果仅

[3] 诺特, Amalie Emmy Noether, 1882.03.23—1935.04.14, 德国数学家。

作空间变换 $\xi_i = 0$, 则有 $I_i = \sum_{\alpha=1}^{s} \eta_{\alpha i} p_\alpha$。

Chapter 2

有心力场

2.1 两体问题

2.1.1 两体问题

作用力可以分类为**有心力 (central force)**与非有心力。有心力的方向永远指向一个固定点,称为力中心点。万有引力、静电力,都是有心力。

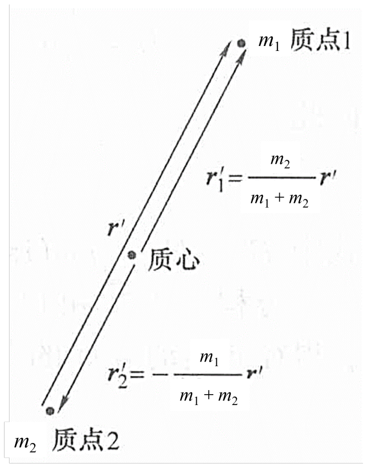

图 2.1: 两体问题

如图 2.1 所示, 两个质点之间的相互作用力是有心力, 力的方向指向另一个质点。设两质点的质量分别为 m_1 和 m_2, $\mathbf{r}'$ 为两质点之间的位矢, $\mathbf{r}_1$ 和 $\mathbf{r}_2$ 分别为两质点相对于质心的位矢, 则有

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}, \tag{2.1.1}$$

从而

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \\ &= \frac{1}{2}m_1\left(\frac{m_2}{m_1 + m_2}\right)^2\dot{\mathbf{r}'}^2 + \frac{1}{2}m_2\left(-\frac{m_1}{m_1 + m_2}\right)^2\dot{\mathbf{r}'}^2 \\ &= \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}'}^2 \end{aligned}$$

$$= \frac{1}{2}\mu \dot{r}'^2, \tag{2.1.2}$$

其中 $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ 或 $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$ 称为**约化质量 (reduced mass)**。因此, 两体问题可以转化为一个质点的运动问题。

两质点相对于质心的动能为

$$T = \frac{1}{2}\mu \dot{r}'^2, \tag{2.1.3}$$

记 $m_c = m_1 + m_2$, 质心相对于坐标原点的位矢 $R = (x_c, y_c, z_c)$, 则有 Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m_c \dot{R}^2 + \frac{1}{2}\mu \dot{r}'^2 - V(r) \\ &= \frac{1}{2}m_c \left(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2\right) + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r), \end{aligned} \tag{2.1.4}$$

因为 L 不显含时间, 故有广义能量

$$H = \frac{1}{2}m_c \left(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2\right) + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r) = E \tag{2.1.5}$$

它和机械能相等。因此, 有相对运动的机械能

$$E' = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r). \tag{2.1.6}$$

由角动量 $p_\phi = \partial L/\partial\dot{\phi} = \mu r^2\dot{\phi}$ 守恒, 可以记

$$r^2\dot{\phi} = h, \tag{2.1.7}$$

则

$$\begin{aligned} E' &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{p_\phi^2}{\mu r^2} + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\mu h^2}{r^2} + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r), \end{aligned} \tag{2.1.8}$$

其中 $V_{\text{eff}}(r) = \dfrac{1}{2}\dfrac{\mu h^2}{r^2} + V(r)$ 称为**有效势能 (effective potential)**, $\dfrac{1}{2}\dfrac{\mu h^2}{r^2}$ 看做惯性离心力 $\mu r\dot{\phi}^2 = \dfrac{\mu h^2}{r^3}$ 的势能。

对式 (2.1.4) 处理, 有

$$\begin{cases} \dfrac{\partial L}{\partial \dot{r}} = \mu\dot{r}, \\ \dfrac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \dfrac{\partial V}{\partial r}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{r}} = \mu\ddot{r}, \end{cases} \tag{2.1.9}$$

记 $F(r) = -\dfrac{\partial V}{\partial r}$, 套用 Lagrange 方程, 得

$$\mu\ddot{r} - \mu r\dot{\phi}^2 - F = 0. \tag{2.1.10}$$

Chapter 3

微振动

3.1 微振动

3.1.1 质量系数与弹性系数

考虑稳定约束下的动能

$$
\begin{aligned}
T = T_2 &= \frac{1}{2}\sum_{i=1}^{n} m_i \sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta \\
&= \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\left(\sum_{i=1}^{n} m_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\cdot\frac{\partial \mathbf{r}_i}{\partial q_\beta}\right)\dot{q}_\alpha\dot{q}_\beta \\
&\approx \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta.
\end{aligned}
\tag{3.1.1}
$$

这里定义了质量系数 $m_{\alpha\beta}$, 其为对称矩阵, 即 $m_{\alpha\beta} = m_{\beta\alpha}$。

定义 3.1: 质量系数

$$
m_{\alpha\beta} = \sum_{i=1}^{n} m_i \left.\frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right|_{q_\alpha=0}\cdot\left.\frac{\partial \mathbf{r}_i}{\partial q_\beta}\right|_{q_\beta=0}
\tag{3.1.2}
$$

为质量系数。

接下来把势能函数 $V = V(q)$ 在平衡位置 $q_\alpha = 0$ 处展开, 得

$$
\begin{aligned}
V &= \cancel{V(0,\dots,0)} + \cancel{\sum_{\alpha=1}^{s}\left.\frac{\partial V}{\partial q_\alpha}\right|_{q_\alpha=0}} q_\alpha + \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\left.\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right|_{q_\alpha=q_\beta=0} q_\alpha q_\beta + \dots \\
&\approx \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} c_{\alpha\beta} q_\alpha q_\beta
\end{aligned}
\tag{3.1.3}
$$

这里定义了弹性系数 $c_{\alpha\beta}$, 其也为对称矩阵, 即 $c_{\alpha\beta} = c_{\beta\alpha}$。

定义 3.2: 弹性系数

$$c_{\alpha\beta} = \left.\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right|_{q_\alpha = q_\beta = 0} \tag{3.1.4}$$

为弹性系数。

动能是广义速度的正定二次型, 势能是广义坐标的正定二次型, 所以 Lagrangian $L = T - V$ 是广义坐标和广义速度的正定二次型, 现在可以写作

$$L = \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_{\alpha\beta}\dot{q}_\alpha \dot{q}_\beta - \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} c_{\alpha\beta} q_\alpha q_\beta. \tag{3.1.5}$$

3.1.2 运动微分方程

由式 (3.1.5), 可得

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}_\gamma} &= \frac{\partial}{\partial \dot{q}_\gamma}\left(\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_{\alpha\beta}\dot{q}_\alpha \dot{q}_\beta\right) \\
&= \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_{\alpha\beta}\delta_{\gamma\alpha}\dot{q}_\beta + \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_{\alpha\beta}\dot{q}_\alpha \delta_{\gamma\beta} \\
&= \frac{1}{2}\sum_{\beta=1}^{s} m_{\gamma\beta}\dot{q}_\beta + \frac{1}{2}\sum_{\alpha=1}^{s} m_{\alpha\gamma}\dot{q}_\alpha \\
&= \frac{1}{2}\sum_{\alpha=1}^{s} m_{\gamma\alpha}\dot{q}_\alpha + \frac{1}{2}\sum_{\alpha=1}^{s} m_{\gamma\alpha}\dot{q}_\alpha \\
&= \sum_{\alpha=1}^{s} m_{\gamma\alpha}\dot{q}_\alpha,
\end{aligned} \tag{3.1.6}$$

再对时间求导, 得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_\gamma}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\alpha=1}^{s} m_{\gamma\alpha}\dot{q}_\alpha\right) = \sum_{\alpha=1}^{s} m_{\gamma\alpha}\ddot{q}_\alpha. \tag{3.1.7}$$

再来看 $\partial L/\partial q_\gamma$, 与求式 (3.1.6) 方法一致, 有

$$\frac{\partial L}{\partial q_\gamma} = -\sum_{\alpha=1}^{s} c_{\gamma\alpha} q_\alpha. \tag{3.1.8}$$

代入 Euler-Lagrange 方程, 得

$$\sum_{\alpha=1}^{s} m_{\gamma\alpha}\ddot{q}_\alpha + \sum_{\alpha=1}^{s} c_{\gamma\alpha} q_\alpha = 0. \tag{3.1.9}$$

Chapter 4

Hamilton 力学

4.1 Hamilton 方程

4.1.1 Legendre 变换

设 $f = f(x, y)$, 则

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y, \tag{4.1.1}$$

记 $u = \frac{\partial f}{\partial x}$, $v = \frac{\partial f}{\partial y}$, 如果把 u, y 看作新的自变量, v, x 看作新的因变量, 那么

$$x = x(u, y), \quad v = v(u, y), \tag{4.1.2}$$

这时函数 f 可以用 u, y 表示, 即 $f = f(x(u, y), y) = F(u, y)$, 则

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} = u\frac{\partial x}{\partial u}, \tag{4.1.3}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y} = u\frac{\partial x}{\partial y} + v, \tag{4.1.4}$$

将式 (4.1.4) 移项, 可得

$$v = -\frac{\partial}{\partial y}(-F + ux) = -\frac{\partial g}{\partial y}, \tag{4.1.5}$$

其中 $g(u, y) = -F + ux = -f + \frac{\partial f}{\partial x}x$。

g 对 u 求偏导数, 可得

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial}{\partial u}(-F + ux) \\ &= -\frac{\partial F}{\partial u} + x + u\frac{\partial x}{\partial u} \\ &= -u\frac{\partial x}{\partial u} + x + u\frac{\partial x}{\partial u} \\ &= x,\end{aligned} \tag{4.1.6}$$

合起来就是

$$\begin{cases} u(x,y) = \dfrac{\partial f(x,y)}{\partial x}, \\ v(x,y) = \dfrac{\partial f(x,y)}{\partial y}, \\ f(x,y) = f(x,y), \end{cases} \implies \begin{cases} x\,(u,y) = \dfrac{\partial g\,(u,y)}{\partial u}, \\ v\,(u,y) = -\dfrac{\partial g\,(u,y)}{\partial y}, \\ g\,(u,y) = u\cdot x\,(u,y) - F\,(u,y)\,, \end{cases} \tag{4.1.7}$$

这种由一组独立变量到另一组独立变量的变换, 称为 **Legendre 变换 (Legendre transformation)**。

4.1.2 Hamilton 方程

前文中定义的式 (1.3.30) $H = \sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - L$, 称为 **Hamiltonian** 。

对于单自由度系统, 将式 (4.1.7) 中的 $x, y, u, v, f(F), g$ 分别替换为 $\dot{q}, q, p, \dot{p}, L, H$, 可得

$$\begin{cases} p\,(\dot{q},q) = \dfrac{\partial L\,(\dot{q},q)}{\partial \dot{q}}, \\ \dot{p}\,(\dot{q},q) = \dfrac{\partial L\,(\dot{q},q)}{\partial q}, \\ L\,(\dot{q},q) = L\,(\dot{q},q)\,, \end{cases} \implies \begin{cases} \dot{q}(p,q) = \dfrac{\partial H(p,q)}{\partial p}, \\ \dot{p}(p,q) = -\dfrac{\partial H(p,q)}{\partial q}, \\ H(p,\dot{q}) = p\cdot\dot{q}(p,q) - L(p,q). \end{cases} \tag{4.1.8}$$

推广一下, 把

$$\boxed{\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}} \tag{4.1.9}$$

称为 **Hamilton 方程 (Hamilton's equations)**或者**正则方程 (canonical equations)**。

4.2 Poisson 括号与 Poisson 定理

4.2.1 Poisson 括号

若 $f = f\,(t,q,p)$ 是相空间的一个函数, 其对时间求导可得

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{s}\left(\frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha\right), \tag{4.2.1}$$

将 Hamilton 方程代入上式, 得

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{s}\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) \\ &= \frac{\partial f}{\partial t} + [f, H]. \end{aligned} \tag{4.2.2}$$

为了简化书写, 引入 **Poisson 括号 (Poisson bracket)**。

定义 4.1

设 f 和 g 都是关于 t, q, p 的函数, 定义 Poisson 括号为

$$[f, g] = \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}. \tag{4.2.3}$$

若 f, g, h 都是关于 t, q, p 的函数, 则有如下性质。

性质 4.1
$[f, f] = 0.$

性质 4.2
$[C, f] = 0$, 其中 C 是常数。

性质 4.3 反对称性
$[f, g] = -[g, f].$

性质 4.4 双线性性
$[Cf, g] = C[f, g] = [f, Cg]$, 其中 C 是常数。且有 $[f, g + h] = [f, g] + [f, h].$

性质 4.5 Leibniz 规则 (Leibniz rule)
$[f, gh] = [f, g]h + g[f, h].$

证明.

$$\begin{aligned}
[f, gh] &= \frac{\partial f}{\partial q_\alpha}\frac{\partial (gh)}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial (gh)}{\partial q_\alpha} \\
&= \frac{\partial f}{\partial q_\alpha}\left(\frac{\partial g}{\partial p_\alpha}h + g\frac{\partial h}{\partial p_\alpha}\right) - \frac{\partial f}{\partial p_\alpha}\left(\frac{\partial g}{\partial q_\alpha}h + g\frac{\partial h}{\partial q_\alpha}\right) \\
&= \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha}h + \frac{\partial f}{\partial q_\alpha}g\frac{\partial h}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}h - \frac{\partial f}{\partial p_\alpha}g\frac{\partial h}{\partial q_\alpha} \\
&= \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right)h + g\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha}\right) \\
&= [f, g]h + g[f, h].
\end{aligned}$$

□

性质 4.6

$$\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right]. \tag{4.2.4}$$

证明.

$$\begin{aligned}\frac{\partial}{\partial t}[f,g] &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right)\\ &= \frac{\partial^2 f}{\partial t\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} + \frac{\partial f}{\partial q_\alpha}\frac{\partial^2 g}{\partial t\partial p_\alpha} - \frac{\partial^2 f}{\partial t\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial^2 g}{\partial t\partial q_\alpha}\\ &= \frac{\partial}{\partial q_\alpha}\left(\frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial p_\alpha} + \frac{\partial f}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}\left(\frac{\partial g}{\partial t}\right) - \frac{\partial}{\partial p_\alpha}\left(\frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial q_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha}\left(\frac{\partial g}{\partial t}\right)\\ &= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right].\end{aligned}$$

□

性质 4.7

$$[q_\alpha, f] = \frac{\partial f}{\partial p_\alpha}, \quad [p_\alpha, f] = -\frac{\partial f}{\partial q_\alpha}. \tag{4.2.5}$$

性质 4.8

$$\left[q_\alpha, q_\beta\right] = \left[p_\alpha, p_\beta\right] = 0, \quad \left[q_\alpha, p_\beta\right] = \delta_{\alpha\beta}. \tag{4.2.6}$$

性质 4.9 Jacobi 恒等式 (Jacobi identity)

$$[f,[g,H]] + [g,[h,f]] + [H,[f,g]] = 0. \tag{4.2.7}$$

Jacobi 恒等式的证明可以参见 Landau 的《力学》一书。其他性质的证明都是很 trival 的，这里就不再赘述了。

根据性质 4.7, 可以把 Hamilton 方程式 (4.1.9) 写成

$$\begin{cases}\dot{q}_\alpha = [q_\alpha, H],\\ \dot{p}_\alpha = [p_\alpha, H].\end{cases} \tag{4.2.8}$$

知道 f 是运动积分的充要条件为 $\mathrm{d}f/\mathrm{d}t = \partial f/\partial t + [f,H] = 0$, 可以考察一下 Hamiltonian:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + [H,H] = \frac{\partial H}{\partial t}, \tag{4.2.9}$$

即得如下推论:

推论 4.1
若 H 不显含时间, 则 H 是运动积分, 系统的 Hamiltonian 守恒。

4.2.2 Poisson 定理

定理 4.1
若力学量 f, g 不显含时间, 则 $[f, g]$ 也不显含时间。

证明. 已知

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, H], \tag{4.2.10}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} + [g, H]. \tag{4.2.11}$$

由性质 4.9, 可得

$$\begin{aligned} 0 &= [H, [f, g]] + [f, [g, H]] + [g, [H, f]] \\ &= [H, [f, g]] + [f, [g, H]] + [g, -[f, H]] \\ &= [H, [f, g]] + \left[f, -\frac{\partial g}{\partial t}\right] + \left[g, \frac{\partial f}{\partial t}\right] \\ &= [[f, g], H] + \left[f, \frac{\partial g}{\partial t}\right] + \left[\frac{\partial f}{\partial t}, g\right] \\ &= [[f, g], H] + \frac{\partial}{\partial t}[f, g] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}[f, g] = 0. \end{aligned} \tag{4.2.12}$$

□

4.3 作用量原理

4.3.1 作用量与作用量原理

定义 4.2: 作用量 (action)

$$S = \int_{t_1}^{t_2} L(t, q, \dot{q})\, \mathrm{d}t. \tag{4.3.1}$$

作用量的变分

$$\delta S = \delta \int_{t_1}^{t_2} L\, \mathrm{d}t = 0 \tag{4.3.2}$$

称为**作用量原理 (action principles)**, 当作用路径两端固定时称为 **Hamilton 原理 (Hamilton's principle)**。

4.3.2 作用量原理的应用

例 4.1
由作用量原理推导 Euler-Lagrange 方程。

解. 由式 (4.3.2), 可得

$$\delta S = \delta \int_{t_1}^{t_2} L \,\mathrm{d}t = 0, \tag{4.3.3}$$

交换变分与积分的次序, 得

$$\int_{t_1}^{t_2} \sum_{\alpha=1}^{s} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha + \frac{\partial L}{\partial q_\alpha} \delta q_\alpha \right) \mathrm{d}t = 0. \tag{4.3.4}$$

又

$$\sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha = \sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \delta q_\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} \delta q_\alpha - \sum_{\alpha=1}^{s} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) \delta q_\alpha, \tag{4.3.5}$$

代入上式可得

$$\int_{t_1}^{t_2} \left[\frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} \delta q_\alpha - \sum_{\alpha=1}^{s} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) \delta q_\alpha + \sum_{\alpha=1}^{s} \frac{\partial L}{\partial q_\alpha} \delta q_\alpha \right] \mathrm{d}t = 0 \tag{4.3.6}$$

$$\Longleftrightarrow \left[\sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} \delta q_\alpha \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{\alpha=1}^{s} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) - \frac{\partial L}{\partial q_\alpha} \right] \delta q_\alpha \,\mathrm{d}t = 0, \tag{4.3.7}$$

又因为端点是固定的, 所以 $\delta q_\alpha(t_1) = \delta q_\alpha(t_2) = 0$, 所以

$$\int_{t_1}^{t_2} \sum_{\alpha=1}^{s} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) - \frac{\partial L}{\partial q_\alpha} \right] \delta q_\alpha \,\mathrm{d}t = 0, \tag{4.3.8}$$

由 δq_α 变分的任意性, 可得

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) - \frac{\partial L}{\partial q_\alpha} = 0. \tag{4.3.9}$$

□

例 4.2
由作用量原理推导 Hamilton 方程。

解. 由 $H = \sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - L$, 可得

$$\delta S = \delta \int_{t_1}^{t_2} (p_\alpha \dot{q}_\alpha - H) \,\mathrm{d}t = 0, \tag{4.3.10}$$

交换变分与积分的次序, 得

$$\begin{aligned}
&\int_{t_1}^{t_2}\sum_{\alpha=1}^{s}\left[\delta\left(p_\alpha\dot{q}_\alpha\right)-\delta H\right]\mathrm{d}t\\
=&\int_{t_1}^{t_2}\sum_{\alpha=1}^{s}\left[\delta p_\alpha\dot{q}_\alpha+p_\alpha\delta\dot{q}_\alpha-\frac{\partial H}{\partial p_\alpha}\delta p_\alpha-\frac{\partial H}{\partial q_\alpha}\delta q_\alpha\right]\mathrm{d}t\\
=&\int_{t_1}^{t_2}\sum_{\alpha=1}^{s}\left[p_\alpha\delta\dot{q}_\alpha+\left(\dot{q}_\alpha-\frac{\partial H}{\partial p_\alpha}\right)\delta p_\alpha-\frac{\partial H}{\partial q_\alpha}\delta q_\alpha\right]\mathrm{d}t\\
=&0,
\end{aligned}\tag{4.3.11}$$

又

$$\begin{aligned}
\sum_{\alpha=1}^{s}p_\alpha\delta\dot{q}_\alpha=&\sum_{\alpha=1}^{s}p_\alpha\frac{\mathrm{d}}{\mathrm{d}t}\delta q_\alpha\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha=1}^{s}p_\alpha\delta q_\alpha-\sum_{\alpha=1}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}p_\alpha\right)\delta q_\alpha\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha=1}^{s}p_\alpha\delta q_\alpha-\sum_{\alpha=1}^{s}\dot{p}_\alpha\delta q_\alpha,
\end{aligned}\tag{4.3.12}$$

代入上式可得

$$\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha=1}^{s}p_\alpha\delta q_\alpha-\sum_{\alpha=1}^{s}\dot{p}_\alpha\delta q_\alpha+\sum_{\alpha=1}^{s}\left(\dot{q}_\alpha-\frac{\partial H}{\partial p_\alpha}\right)\delta p_\alpha-\sum_{\alpha=1}^{s}\frac{\partial H}{\partial q_\alpha}\delta q_\alpha\right]\mathrm{d}t=0\tag{4.3.13}$$

$$\Longleftrightarrow\ \sum_{\alpha=1}^{s}p_\alpha\delta q_\alpha\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\sum_{\alpha=1}^{s}\left[\left(\dot{q}_\alpha-\frac{\partial H}{\partial p_\alpha}\right)\delta p_\alpha-\left(\dot{p}_\alpha+\frac{\partial H}{\partial q_\alpha}\right)\delta q_\alpha\right]\mathrm{d}t=0,\tag{4.3.14}$$

又因为端点是固定的, 所以 $\delta q_\alpha(t_1)=\delta q_\alpha(t_2)=0$, 所以

$$\int_{t_1}^{t_2}\sum_{\alpha=1}^{s}\left[\left(\dot{q}_\alpha-\frac{\partial H}{\partial p_\alpha}\right)\delta p_\alpha-\left(\dot{p}_\alpha+\frac{\partial H}{\partial q_\alpha}\right)\delta q_\alpha\right]\mathrm{d}t=0,\tag{4.3.15}$$

由 $\delta q_\alpha,\delta p_\alpha$ 变分的任意性, 可得

$$\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha},\quad \dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}.\tag{4.3.16}$$

□

4.4 正则变换

4.4.1 正则变换的条件

定义 4.3: 正则变换

设 P_α 和 Q_α 是关于 t, p_α, q_α 的函数。若能找到一个新函数 K, 使得 Hamilton 方程的形式不变, 即

$$\dot{Q}_\alpha = \frac{\partial K}{\partial P_\alpha}, \quad \dot{P}_\alpha = -\frac{\partial K}{\partial Q_\alpha}, \tag{4.4.1}$$

那么, 称 Q_α, P_α 是 q_α, p_α 的**正则变换 (canonical transformation)**。K 是正则变换后新的 Hamiltonian。

由作用量原理可得, 正则变换的条件为

$$\delta \int_{t_1}^{t_2} \left(\sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - H \right) \mathrm{d}t = \delta \int_{t_1}^{t_2} \left(\sum_{\alpha=1}^{s} P_\alpha \dot{Q}_\alpha - K \right) \mathrm{d}t = 0, \tag{4.4.2}$$

不妨加上一个任意函数 U 对时间的导数, 即

$$\sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - H = \sum_{\alpha=1}^{s} P_\alpha \dot{Q}_\alpha - K + \frac{\mathrm{d}U}{\mathrm{d}t}, \tag{4.4.3}$$

把 U 称为正则变换的**母函数 (generating function)**或**生成函数 (generating function)**。即有

$$\mathrm{d}U = \sum_{\alpha=1}^{s} p_\alpha \, \mathrm{d}q_\alpha - \sum_{\alpha=1}^{s} P_\alpha \, \mathrm{d}Q_\alpha + (K - H) \, \mathrm{d}t. \tag{4.4.4}$$

4.4.2 四型正则变换

1. $U_1 = U_1(t, q, Q)$, 称为**第一型正则变换 (type 1 canonical transformation)**。
2. $U_2 = U_2(t, q, P)$, 称为**第二型正则变换 (type 2 canonical transformation)**。
3. $U_3 = U_3(t, p, Q)$, 称为**第三型正则变换 (type 3 canonical transformation)**。
4. $U_4 = U_4(t, p, P)$, 称为**第四型正则变换 (type 4 canonical transformation)**。

第一型正则变换

$$\mathrm{d}U_1 = \sum_{\alpha=1}^{s} \frac{\partial U_1}{\partial q_\alpha} \mathrm{d}q_\alpha + \sum_{\alpha=1}^{s} \frac{\partial U_1}{\partial Q_\alpha} \mathrm{d}Q_\alpha + \frac{\partial U_1}{\partial t} \mathrm{d}t. \tag{4.4.5}$$

令 $U_1 = U$, 比较式 (4.4.4) 和 (4.4.5), 可得

$$\begin{cases} p_\alpha = \dfrac{\partial U_1}{\partial q_\alpha}, \\ P_\alpha = -\dfrac{\partial U_1}{\partial Q_\alpha}, \\ K = H + \dfrac{\partial U_1}{\partial t}. \end{cases} \tag{4.4.6}$$

第二型正则变换

$$\mathrm{d}U_2 = \sum_{\alpha=1}^{s} \frac{\partial U_2}{\partial q_\alpha} \mathrm{d}q_\alpha + \sum_{\alpha=1}^{s} \frac{\partial U_2}{\partial P_\alpha} \mathrm{d}P_\alpha + \frac{\partial U_2}{\partial t} \mathrm{d}t. \tag{4.4.7}$$

令 $U_2 = U + \sum_{\alpha=1}^{s} P_\alpha Q_\alpha$, 则

$$\mathrm{d}U_2 = \sum_{\alpha=1}^{s} p_\alpha \mathrm{d}q_\alpha - \cancel{\sum_{\alpha=1}^{s} P_\alpha \mathrm{d}Q_\alpha} + (K - H)\,\mathrm{d}t + \cancel{\sum_{\alpha=1}^{s} P_\alpha \mathrm{d}Q_\alpha} + \sum_{\alpha=1}^{s} Q_\alpha \mathrm{d}P_\alpha. \tag{4.4.8}$$

比较式 (4.4.4) 和 (4.4.8), 可得

$$\begin{cases} p_\alpha = \dfrac{\partial U_2}{\partial q_\alpha}, \\ Q_\alpha = \dfrac{\partial U_2}{\partial P_\alpha}, \\ K = H + \dfrac{\partial U_2}{\partial t}. \end{cases} \tag{4.4.9}$$

第三型正则变换

$$\mathrm{d}U_3 = \sum_{\alpha=1}^{s} \frac{\partial U_3}{\partial p_\alpha} \mathrm{d}p_\alpha + \sum_{\alpha=1}^{s} \frac{\partial U_3}{\partial Q_\alpha} \mathrm{d}Q_\alpha + \frac{\partial U_3}{\partial t} \mathrm{d}t. \tag{4.4.10}$$

令 $U_3 = U - \sum_{\alpha=1}^{s} p_\alpha q_\alpha$, 则

$$\mathrm{d}U_3 = \cancel{\sum_{\alpha=1}^{s} p_\alpha \mathrm{d}q_\alpha} - \sum_{\alpha=1}^{s} P_\alpha \mathrm{d}Q_\alpha + (K - H)\,\mathrm{d}t - \cancel{\sum_{\alpha=1}^{s} p_\alpha \mathrm{d}q_\alpha} - \sum_{\alpha=1}^{s} q_\alpha \mathrm{d}p_\alpha. \tag{4.4.11}$$

比较式 (4.4.4) 和 (4.4.11), 可得

$$\begin{cases} q_\alpha = -\dfrac{\partial U_3}{\partial p_\alpha}, \\ P_\alpha = -\dfrac{\partial U_3}{\partial Q_\alpha}, \\ K = H + \dfrac{\partial U_3}{\partial t}. \end{cases} \tag{4.4.12}$$

第四型正则变换

$$\mathrm{d}U_4 = \sum_{\alpha=1}^{s} \frac{\partial U_4}{\partial p_\alpha} \mathrm{d}p_\alpha + \sum_{\alpha=1}^{s} \frac{\partial U_4}{\partial P_\alpha} \mathrm{d}P_\alpha + \frac{\partial U_4}{\partial t} \mathrm{d}t. \tag{4.4.13}$$

令 $U_4 = U - \sum_{\alpha=1}^{s} p_\alpha q_\alpha + \sum_{\alpha=1}^{s} P_\alpha Q_\alpha$, 则

$$\mathrm{d}U_4 = \cancel{\sum_{\alpha=1}^{s} p_\alpha \,\mathrm{d}q_\alpha} - \cancel{\sum_{\alpha=1}^{s} P_\alpha \,\mathrm{d}Q_\alpha} + (K - H)\,\mathrm{d}t - \cancel{\sum_{\alpha=1}^{s} p_\alpha \,\mathrm{d}q_\alpha} - \sum_{\alpha=1}^{s} q_\alpha \,\mathrm{d}p_\alpha + \cancel{\sum_{\alpha=1}^{s} P_\alpha \,\mathrm{d}Q_\alpha} + \sum_{\alpha=1}^{s} Q_\alpha \,\mathrm{d}P_\alpha. \tag{4.4.14}$$

比较式 (4.4.4) 和 (4.4.14), 可得

$$\begin{cases} q_\alpha = -\dfrac{\partial U_4}{\partial p_\alpha}, \\ Q_\alpha = \dfrac{\partial U_4}{\partial P_\alpha}, \\ K = H + \dfrac{\partial U_4}{\partial t}. \end{cases} \tag{4.4.15}$$

四型正则变换的总结

表 4.1: 四型正则变换

类别	生成函数	正则变换
第一类	$U = U(t, q, Q)$	$p = \partial U/\partial q, \quad P = -\partial U/\partial Q$
第二类	$U = U(t, q, P)$	$p = \partial U/\partial q, \quad Q = \partial U/\partial P$
第三类	$U = U(t, p, Q)$	$q = -\partial U/\partial p, \quad P = -\partial U/\partial Q$
第四类	$U = U(t, p, P)$	$q = -\partial U/\partial p, \quad Q = \partial U/\partial P$

注：$K = H + \partial U/\partial t$。

附录 A

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索引